ON THE STABILITY OF HOMOGENEOUS STATES

(OB USTOICHIVOSTI ODNORODNYKH SOSTOIANII)

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The problem of determining stability criteria for systems weakly dependent on spatial variables has been the subject of many recent papers (see [1] and surveys [2 and 3]). The systems considered in these papers are such, that instability is in fact manifested in the finite region of space, while the boundary conditions require that the solutions tend to zero at infinity. These stability criteria take the form of a quantization condition which, in the case of two variables x and t, can be written as [1]

$$\int [k_i(\omega, x) - k_j(\omega, x)]\, dx = \pi n \tag{1}$$

from which one can determine ω. The solutions corresponding to values of ω found in this manner are called quasi-classical. Here k_i and k_j are the roots of the dispersion equation (in which x is regarded as a parameter), n is a whole number, and the integration is performed in the complex plane x between the branch points x_1 and x_2 at which $k_i = k_j$, of the many-valued analytic function $K(\omega, x)$. However, as shown in [4] which contains a detailed investigation of a single second-order equation, fulfillment of condition (1) does not always guarantee the existence of an eigenfunction damped at infinity. Paper [4] gives the conditions necessary for the existence of an eigenfunction to follow from the fulfillment of equation (1). These conditions are connected with the topological structure of the Stokes lines on the complex plane x, and are difficult to verify even in the case of a second-order equation.

Considered below are the systems dependent on two variables x and t. It is assumed that the state being tested for stability is homogeneous and independent of time, and that the points $x = \pm L$ at which the boundary conditions are stipulated are sufficiently far apart (the asymptotic form of the stability condition is considered for $L \to \infty$). Investigation of this case, which is simpler than the one dealt with in [1 to 4], can be carried out for arbitrary systems. Below it is shown that for sufficiently large L there generally exist two types of non-trivial solutions of the boundary value problem: 'one-sided' solutions, for which the complex frequency ω is determined by the boundary conditions at one of the ends, and 'global' solutions for which ω is independent of the specific form of the boundary conditions. Global solutions are nalogous to the quasiclassical solutions found for the weakly inhomogeneous case, but with specific and definite i and j in Equation (1).

It is shown that for the global instability (i.e. the existence of global solutions with $\text{Im}\,\omega > 0$) the existence of ω with $\text{Im}\,\omega > 0$ such that $\text{Im}\,k(\omega) = 0$ for at least one branch of the multi-valued function $k(\omega)$ is necessary, but not sufficient. The relationship between global instability and the absolute instability of the unbounded problem is considered.

The study of the stability of a homogeneous time-independent state requires consideration of a linearized system of equations. We shall assume this to be a system of partial differential equations (the possibility of extending the results to other cases will also be examined) whose coefficients are constant in the present case. We shall write this system in the form

$$\sum_{j=1}^{n} P_{ij}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) u_j = 0 \qquad (i, j = 1, \ldots, n) \tag{2}$$

where P_{ij} are polynomials in $\partial/\partial t$ and $\partial/\partial x$, and $u_j(t, x)$ are unknown functions. The (homogeneous) boundary conditions are written as

$$\sum_{j=1}^{n} \left[B_{\alpha j}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) u_j(t, x)\right]_{x=-L} = 0, \qquad \sum_{j=1}^{n} \left[B_{\beta j}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) u_j(t, x)\right]_{x=L} = 0 \tag{3}$$

where $B_{\alpha j}$ and $B_{\beta j}$ are polybomials in $\partial/\partial t$ and $\partial/\partial x$.

We shall assume that the condition of Petrovskii [5] is fulfilled for system (2), i.e. that the dispersion equation

$$|P_{ij}(-i\omega, ik)| = 0 \tag{4}$$

(which results from the requirement that the determinant of system (2) be equal to zero when the solution is sought in the form $\exp i(kx - \omega t)$) is such, that for all real k the values of ω satisfying equation (4) satisfy the inequality $\text{Im}\,\omega < M$, i.e. the rate of growth of the sinusoidal perturbations is bounded. This condition will be fulfilled by all the correctly stated problems.

We shall assume that boundary conditions (3) are independent of the equations and of each other, i.e. that none of them are satisfied by virtue of the equations and the remaining boundary conditions (none are identically satisfied on the set of solutions of equations (2) subject to the remaining boundary conditions). We shall say that a wave of the form $\exp i\,[k_m(\omega)x - \omega t]$, given by the branch $k_m(\omega)$ proceeds in a certain direction if for $\text{Im}\,\omega > M$ it diminishes with change of x in this direction. Thus, if the inequality $\text{Im}\,k_i > 0$ is fulfilled for $\text{Im}\,\omega > M$, then we shall say that the i-th wave is moving in the direction to the right, while if $\text{Im}\,k_j < 0$, then the j-th wave is moving in the direction to the left. The condition of correctness of the boundary value problem under consideration as developed in [6] can then be formulated as follows. The number of boundary conditions on each end of a segment is equal to the number of waves emanating from it, so as to guarantee the existence of an M_1 such, that for ω with $\operatorname{Im}\omega > \max(M, M_1)$ the amplitudes of the emanating waves are uniquely determined by the amplitude of the incident waves. By virtue of the Petrovskii condition for $\operatorname{Im}\omega > M$ equation (4) has no real roots k, so that every wave propagates either to the right or to the left. At the same time, the total number of boundary conditions (3) must equal the total number of waves, which we denote by N. Denoting the number of waves proceeding to the right by s, we have, in equations (3), $\alpha = 1, \ldots, s$, $\beta = s+1, \ldots, N$.

Let us try to find the eigenfunctions of the problem (2) and (3). Choosing some ω, we can write the general solution of equations (2) as

$$u_j(t, x) = v_j(\omega, x)\, e^{-i\omega t}, \qquad v_j(\omega, x) = \sum_{l=1}^{N} C_l w_{jl}(\omega)\, e^{ik_l(\omega)x} \tag{5}$$

where C_l are arbitrary constants. Substituting these expressions into the boundary conditions (3), we have

$$\begin{gathered}\sum_{l=1}^{N} C_l a_{\alpha l}(\omega)\, e^{-ik_l(\omega)L} = 0 \qquad (\alpha = 1, \ldots, s)\\ \sum_{l=1}^{N} C_l a_{\beta l}(\omega)\, e^{ik_l(\omega)L} = 0 \qquad (\beta = s+1, \ldots, N)\end{gathered} \tag{6}$$

where

$$a_{\alpha l}(\omega) = \sum_{j=1}^{n} B_{\alpha j}\,[-i\omega, ik_l(\omega)]\, w_{jl}(\omega), \qquad a_{\beta l}(\omega) = \sum_{j=1}^{n} B_{\beta j}\,[-i\omega, ik_l(\omega)]\, w_{jl}(\omega)$$

The necessary condition for the existence of an eigenfunction is that the determinant of system (6) used for finding the constants C_l vanishes.

Let us examine the asymptotic behavior of the roots ω of this determinant as $L \to \infty$. For each ω not lying on any of the curves $\operatorname{Im}[k_i(\omega) - k_j(\omega)] = 0$, we shall arrange the $k_i(\omega)$ in the order of decreasing of their imaginary parts.

$$\operatorname{Im} k_1 > \operatorname{Im} k_2 > \cdots \operatorname{Im} k_N \tag{7}$$

At $L \to \infty$, the principal term in the determinant of system (6) contains the largest exponential factor

$$E = \exp i\Big[\sum_{l=1}^{s} k_l(\omega) - \sum_{l=s+1}^{N} k_l(\omega)\Big] L \tag{8}$$

multiplied by the product of the determinants of two minors of the matrix $A = \|a_{ij}\|$. One of them, A_s, is of the order s and occupies the upper left-hand corner of the matrix A, while the other, A_{N-s}, is of the order N-s and occupies the lower right-hand corner of the matrix A. The term of the next order of smallness is of the order $\exp i\,[-2\,(k_s - k_{s+1})]\,L$, with respect to the first, and the coefficient of the exponential is the product of the determinants of the similarly situated minors A'_s and A'_{N-s} of the matrix obtained from A by the interchange of the s-th and $s+1$-th columns. Thus, retaining the two principal terms in the determinant of system (6), we have

$$D = |A_s|\,|A_{N-s}| + |A_s'|\,|A'_{N-s}|\exp\{-2i\,[k_s(\omega) - k_{s+1}(\omega)]L\} \qquad (9)$$

where D denotes the determinant of system (6) divided by E.

If $\operatorname{Im}(k_s - k_{s+1}) \neq 0$, then the fact that the determinant D is equal to zero means, that

$$|A_s|\,|A_{N-s}| = 0 \qquad (10)$$

If the boundary value problem is indeed correct, then equation (9) is not identically satisfied with respect to ω, since this would mean, in particular, that for ω with $\operatorname{Im}\omega > \max(M, M_1)$ the amplitudes of the waves moving away from the boundaries cannot be determined uniquely. Consideration of the next term in equation (9) can contribute only slight corrections to ω found from (10). Equation (10) means that as $L \to \infty$, system of equations (6) breaks down into two independent subsystems, coefficients of each of which depend on the boundary conditions at one end only. If just one of the determinants in equation (10) is equal to zero, then only those C_l are different from zero which correspond to the waves that die out most rapidly as they move away from the end on which the boundary conditions entering into this determinant are given. If both determinants are simultaneously equal to zero, then two solutions arise, which are subject to the boundary conditions at different ends and independent of each other.

It is natural to call the solutions associated with the vanishing of the determinants $|A_s|$ and $|A_{N-s}|$ 'one-sided'. The possibility of the existence of one-sided solutions independent of the boundary conditions at the other end can be explained physically by the fact for $\operatorname{Im}(k_s - k_{s+1}) \neq 0$ the amplitudes of the waves reflected from the other end tend to zero as $L \to \infty$ in the neighborhood of the end under consideration.

In addition to the case considered, it is also possible to have a situation where both terms in the right-hand side of equation (9) are of the same order and add up to zero. This can occur if the inequality

$$\operatorname{Im}[k_s(\omega) - k_{s+1}(\omega)] = 0 \qquad (11)$$

is satisfied at the limit as $L \to \infty$.

Here and henceforth we shall be concerned with the general case when the product $|A_s'|\,|A_{N-s}'|$ is not identically equal to zero with respect to ω. This can always be achieved by slight alteration in the boundary conditions. Since the number s in the boundary conditions is, for $x = -L$, equal to the number of waves moving to the right, it follows that the inequalities $\operatorname{Im} k_s(\omega) > 0$ and $\operatorname{Im} k_{s+1}(\omega) < 0$ are fulfilled for $\operatorname{Im}\omega > M$, so that the wave corresponding to $k_s(\omega)$ moves to the right, while the wave corresponding to $k_{s+1}(\omega)$ moves to the left. If Equation (11) is fulfilled for some value of ω, then

$$k_s(\omega) - k_{s+1}(\omega) \neq \text{const} \tag{12}$$

since Equation (11) is obviously not fulfilled for Im $\omega > M$. At the same time, since equation (11) cannot be fulfilled for any two-dimensional portion of the plane ω, Equation (11) is the equation of some curve on the plane ω. In fact, by virtue of the analyticity of the functions $k_s(\omega)$ and $k_{s+1}(\omega)$ this would mean that the equation (11) would be identically satisfied for all ω, which is impossible by virtue of (12). If Equation (11) is fulfilled for some ω_0, then for a sufficiently large L it is possible to find in its neighborhood such $\omega = \omega_0 + \Delta\omega$, which would make the entire right-hand side of equation (9) vanish. Indeed, it can be assumed that the quantities $|A_s'|\,|A'_{N-s}|$ and $\frac{\partial}{\partial\omega}[k_s - k_{s+1}]$ are not equal to zero at the point ω_0. If this were not so, then it would always be possible to find another arbitrarily close point where these conditions were fulfilled. Then, making use of the fact that L is large and neglecting terms on the order of $\Delta\omega$ i.e. small compared with $L\,\Delta\omega$, we obtain from the condition of vanishing of the right-hand side of Equation (9), the following equation for $\Delta\omega$:

$$\exp\left\{-2iL\Delta\omega\frac{\partial}{\partial\omega}[k_s(\omega) - k_{s+1}(\omega)]\right\} = -\frac{|A_s|\,|A_{N-s}|}{|A_s'|\,|A_{N-s}'|}\exp[2i(k_s - k_{s+1})] \tag{13}$$

This equation has a set of solutions situated near the curve (11) in the neighborhood of the point ω_0. Consideration of terms of the next order of smallness not appearing in the right-hand side of (9) yields negligible corrections in the value of ω.

Thus, (11) as well as (10) can be regarded as the limiting form of the equation used for determining the eigenvalues of ω. Since the determination of the eigenfunctions corresponding to the eigenvalues of (11) requires that not only the minors corresponding to the minors A_s and A_{N-s} of the matrix A, but also the neighboring columns be retained in the determinant of system (6), the mechanism of constructing the eigenfunction can be presented as follows.

Waves corresponding to $k_1, \ldots, k_s$ are excited at the end $x = -L$ for some frequency ω. When these waves arrive at the end $x = L$, one of them, namely the s-th wave has for large L, an amplitude which exceeds considerably the amplitudes of all the other waves. The latter therefore need not be taken into account in seeking the amplitudes of the waves reflected at $x = L$ and corresponding to $k_{s+1}, \ldots, k_N$. When these waves arrive at the point $x = -L$, analogous considerations allow us to disregard all of the incident waves except the $(s+1)$-th in determining the amplitudes of the waves reflected at this end. In order for an eigenfunction to exist, it is necessary that the reflection of the $(s+1)$-th wave at the point $x = -L$ results in the formation of an s-th wave possessing the initial amplitude. Equation (11) is an approximate expression of this condition.

It is easy to understand the physical meaning of the requirement that $|A_s'||A'_{N-s}| \neq 0$. It is simply that when the s-th wave is reflected at the point $x = L$ there forms and $(s + 1)$-th wave of non-zero amplitude, and when the $(s + 1)$-th wave is reflected at the point $x = -L$ and s-th wave of non-zero amplitude is formed. Since the solutions under consideration involve the passage of the corresponding waves over the entire segment $[-L, L]$, we call them global (to distinguish them from the one-sided solutions considered above). The form of equation (11) indicates that the stability criteria based on this equation are rather the outcome of the nature of the system in question, than of the specific form of boundary conditions. Comparing this equation with (1), one concludes that the global solutions constitute an analogue to the quasiclassical ones. Suffice it to say that in Equation (11), fully specific branches of the function $k(\omega)$ occur.

Let us now consider the matter of stability as it relates to equation (11). A sufficient condition for the instability to occur is, that at least a portion of the curve given by equation (11) lies in the upper half-plane of the plane ω. Otherwise, there is no global instability. It can be shown that in the presence of global instability one can find values of ω such that

$$\operatorname{Im} \omega > 0, \quad \operatorname{Im} k(\omega) = 0 \tag{14}$$

for at least one branch of $k(\omega)$, i.e. a condition often regarded as the condition of instability of unbounded systems (e.g. see [7]) is fulfilled.

Let us consider ω with large values of Im ω. Here the roots of $k_j(\omega)$ fall into two groups: an upper group $k_1, \ldots, k_s$ with Im $\omega > 0$ and a lower group $k_{s+1}, \ldots, k_N$ with Im $\omega < 0$. If Im ω decreases, then in the case of global instability one can find for some Im $\omega = \beta > 0$ a point ω_* for which relations (11) hold. Since k_s and k_{s+1} were originally on the opposite sides of the real axis k, either, one of them would have had to intersect the real axis, or, Im $k_s =$ Im $k_{s+1} = 0$ for $\omega = \omega_*$. Thus, fulfillment of (11) implies, for Im $\omega > 0$, (14). It is obvious that the converse statement is generally not true. Such a case was, for example, considered in [8], where a distinction was discovered between the stability criterion obtained for a certain real system by satisfying the conditions of the boundary value problem at $L \to \infty$ exactly, and the criteria which follows from (14).

Theres exists an important class of problems where the fulfillment of relations (14) makes it possible to decide the whether or not the global instability is present. These problems involve equations invariant under the substitution of x by $-x$. In this case each ω has a corresponding $-k$ as well as a k and (14) implies the fulfillment of equation (11) for the same ω as in (14).

Let us now look into the connection between global instability and absolute instability (of the unbounded problem), which involves the existence of branch points of the function $k(\omega)$ in the upper half-plane ω and the fulfillment of conditions (14) for one of the branches intersecting at this point. It is clear that if the branches meeting at the branch point include some belonging to different groups ($k_1, \ldots, k_s$ and $k_{s+1}, \ldots, k_N$), then equations (11) are fulfilled and instability exists.

The considerations developed above can be applied not only to systems involving partial differential equations in two independent variables, but also to systems in which an infinite number of values of k_i (ω) correspond to each ω. The classic example of such a problem is the problem of the stability of flow of a viscous incompressible fluid through a tube of a constant cross-section [7], where the relationship k (ω) is determined from the condition of existence of a non-trivial solution of the boundary-value problem for the Orr-Sommerfeld equation. As in the case considered above, for sufficiently large Im ω the roots of k (ω) fall into two groups: an upper group with Im $k > 0$ and a lower group with Im $k < 0$. The condition that these two groups remain separated by a strip parallel to the real axis k as Im ω diminishes to zero, is sufficient for the absence of global instability. If, on the other hand, such a strip does not exist, then there may be an ω for which equation (11) is fulfilled. In this case k_s (ω) must be regarded as the root of the upper group with the least Im k (ω), and k_{s+1} (ω) as the root of the lower group with the largest Im k (ω). This involves the appearance of global instability.

All of the conclusions of the present study can also be applied to the investigation of the stability of states inhomogeneous with respect to x, where the inhomogeneity is confined to zones narrow compared with L, e.g. in the neighborhood of the ends of the segment. The boundary conditions at the ends of the segment together with the adjacent regions of inhomogeneity generate certain effective boundary conditions which can be stipulated at the points lying in the fundamental region where the non-perturbed state is homogeneous, and which relate the amplitudes of the incident waves from the fundamental region to the amplitudes of the waves reflected back into the fundamental region.

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